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Integrability aspects of a Schwarzian PDE

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Abstract

We present a reduction of the anti-self-dual Yang–Mills (ASDYM) equations to a system of partial differential equations (PDEs) introduced recently by Nijhoff et al. (Phys. Lett. A 267 (2000) 147). An auto-Bäcklund transformation of the reduced system is also presented. The system under consideration is related to a fourth-order nonlinear PDE of the Schwarzian type. The symmetry group of the latter equation is calculated and similarity reductions to the Schwarzian equation and the full Painlevé III, V and VI are presented. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

A novel class of integrable nonlinear PDEs of the Schwarzian type, i.e., invariant under projective transformations of the dependent variable, was recently introduced in [1]. The members of this class can be classified into three groups of two-parameter families of equations, named in [1] the Schwarzian PDE, the modified PDE and the regular PDE, respectively. These groups form a Miura chain, and their importance stems from the fact that they are generating equations for whole hierarchies of higher-order integrable systems.

In this Letter we expand on the integrability aspects of the last group of the Nijhoff–Hone–Joshi chain of equations, i.e., on the regular PDE (RPDE). It is well known that a (partial) list of characteristics of integrable systems consists of Lax pairs, Bäcklund transformations, multi-Hamiltonian structure, and reductions to the Painlevé type of ordinary differential equations (ODEs). More recently, one more characteristic has emerged, namely that (most) integrable systems result from reductions of the anti-self-dual Yang–Mills equations (ASDYM) [2,3]. In a sense, one can now consider the reduction *to* Painlevé type equations and the reductions *from* the ASDYM equations as establishing integrability from below and above, respectively. In the following pages we establish the integrability of the RPDE from above, as well as from below. Moreover, we present a Lax pair and a Bäcklund transformation, the latter of which allows for an, essentially, algebraic way of producing new solutions of the RPDE from known, simpler ones.

The reduction of the ASDYM equations which leads to the RPDE is presented in Section 2. More specifically, in that section we show that a two-dimensional reduction, which employs a pair of commuting, null conformal Killing vectors, leads to the system of

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PDEs (25)–(27) for a triad of complex-valued functions (P, R, U). This system expresses the RPDE in involution form, in a sense that will be clear later.

The similarity reductions of the RPDE are presented in Section 4. More specifically, after a thorough analysis of the Lie point symmetries of the RPDE, we show, in Section 4, that the most general similarity reductions of the RPDE lead to the Schwarzian equation and to Painlevé III, V and VI in full form.

The reduction process that gave system (25)–(27) starting from the ASDYM equations has allowed us to construct an auto-Bäcklund transformation of the above system, and, a fortiori, of the RPDE. This Bäcklund transformation, along with the corresponding Bianchi commuting diagram, is presented in Section 3.

The Letter concludes with the perspectives, where, in the framework of a more general program of investigation in progress, we relate the RPDE to the famous Ernst equation of general relativity.

2. Reduction of the ASDYM equations

2.1. General considerations

Throughout this section we shall follow the notation and conventions of [4]. Let $\mathbb{U}$ denote the manifold $\mathbb{R}^4$ with double null coordinates $x^a = (w, z, \tilde{w}, \tilde{z})$ and the metric

$$ds^2 = 2(dz\,d\tilde{z} - dw\,d\tilde{w}) \tag{1}$$

with signature $++--$ (the ultrahyperbolic slice). The Yang–Mills potential on $\mathbb{U}$ is represented by the one-form

$$\Phi = \Phi_w\,dw + \Phi_z\,dz + \Phi_{\tilde{w}}\,d\tilde{w} + \Phi_{\tilde{z}}\,d\tilde{z},$$

where the components take values in the Lie algebra $\mathfrak{g}$ of a Lie group G called the gauge group. In the finite-dimensional case G can be taken to be $\mathbf{GL}(N, \mathbb{C})$. The field strength F of Φ is the two-form given by

$$F = d\Phi + \Phi \wedge \Phi, \tag{2}$$

or, in component form,

$$F_{ab} = \partial_a\Phi_b - \partial_b\Phi_a + [\Phi_a, \Phi_b].$$

We say that Φ and Φ' are related by a gauge transformation when

$$\Phi' = g^{-1}\Phi g + g^{-1}\,dg, \tag{3}$$

where $g(x^a) \in G$.

Φ is said to be anti-self-dual iff F is Hodge anti-self-dual with respect to metric (1). Choosing an orientation, this condition is equivalent to the ASDYM equations

$$\partial_z\Phi_w - \partial_w\Phi_z + [\Phi_z, \Phi_w] = 0, \tag{4}$$

$$\partial_{\tilde{z}}\Phi_{\tilde{w}} - \partial_{\tilde{w}}\Phi_{\tilde{z}} + [\Phi_{\tilde{z}}, \Phi_{\tilde{w}}] = 0, \tag{5}$$

$$\partial_z\Phi_{\tilde{z}} - \partial_{\tilde{z}}\Phi_z - \partial_w\Phi_{\tilde{w}} + \partial_{\tilde{w}}\Phi_w + [\Phi_z, \Phi_{\tilde{z}}] - [\Phi_w, \Phi_{\tilde{w}}] = 0. \tag{6}$$

These equations are the integrability conditions of the overdetermined linear system (Lax pair)

$$\big(\partial_w + \Phi_w - \zeta(\partial_{\tilde{z}} + \Phi_{\tilde{z}})\big)\Psi = 0, \tag{7}$$

$$\big(\partial_z + \Phi_z - \zeta(\partial_{\tilde{w}} + \Phi_{\tilde{w}})\big)\Psi = 0, \tag{8}$$

for all values of the parameter ζ, where $\Psi(x^a; \zeta)$ is a G-valued function of the spacetime coordinates and the spectral parameter.

2.2. Reduced equations and Lax pair

To perform a two-dimensional reduction one has to first choose a two-dimensional subgroup H of the full group of conformal transformations of the ASDYM equations. A general class of two-dimensional reductions is considered in [4] where H is generated by two conformal Killing vectors (CKV)

$$X = a\partial_w + b\partial_z + \tilde{a}\partial_{\tilde{w}} + \tilde{b}\partial_{\tilde{z}},$$
$$Y = c\partial_w + d\partial_z + \tilde{c}\partial_{\tilde{w}} + \tilde{d}\partial_{\tilde{z}}, \tag{9}$$

where a, b, c, d and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ depend only on w, z and $\tilde{w}, \tilde{z}$, respectively. Moreover, the quadruples $\{X, Y, \partial_w, \partial_z\}$ and $\{X, Y, \partial_{\tilde{w}}, \partial_{\tilde{z}}\}$ should both be linearly independent and the reduced metric on the orbits of H should be nondegenerate. These conditions assure a compatible reduction.

Let us consider the following commuting, null CKVs of the above family:

$$X = w\partial_w + \tilde{z}\partial_{\tilde{z}}, \qquad Y = z\partial_z + \tilde{w}\partial_{\tilde{w}}. \tag{10}$$

The invariant spacetime coordinates on the orbits of the two-dimensional group of conformal transformations generated by X, Y are arbitrary functions of the fractions $w/\tilde{z}, z/\tilde{w}$. Without loss of generality we choose them to be

$$u = \frac{w}{\tilde{z}}, \qquad v = \frac{z}{\tilde{w}}. \tag{11}$$

One can show that the reduced metric on the orbits of H generated by X, Y is conformal to two-dimensional Minkowski spacetime in null coordinates, i.e.,

$$ds^2 = \frac{2}{v-u}\,du\,dv. \tag{12}$$

The invariance conditions of the potential Φ, with respect to the algebra generated by X, Y, are

$$\mathcal{L}_X\Phi = \mathcal{L}_Y\Phi = 0, \tag{13}$$

where $\mathcal{L}_X$ denotes the Lie derivative along X. Under these conditions the components of the gauge potential one-form Φ become

$$\Phi_w = \frac{1}{w}A(u,v), \qquad \Phi_z = \frac{1}{z}B(u,v),$$
$$\Phi_{\tilde{w}} = \frac{1}{\tilde{w}}\tilde{A}(u,v), \qquad \Phi_{\tilde{z}} = \frac{1}{\tilde{z}}\tilde{B}(u,v). \tag{14}$$

The reduced ASDYM equations are

$$vA_{,v} - uB_{,u} + [B,A] = 0, \tag{15}$$
$$v\tilde{B}_{,v} - u\tilde{A}_{,u} + [\tilde{B},\tilde{A}] = 0, \tag{16}$$
$$u\big(v\tilde{B}_{,v} + uB_{,u} + [B,\tilde{B}]\big) - v\big(u\tilde{A}_{,u} + vA_{,v} + [A,\tilde{A}]\big) = 0. \tag{17}$$

It is possible to choose a gauge where $A = B = 0$ whereby system (15)–(17) becomes

$$v\tilde{B}_{,v} - u\tilde{A}_{,u} + [\tilde{B},\tilde{A}] = 0, \tag{18}$$
$$\tilde{B}_{,v} - \tilde{A}_{,u} = 0. \tag{19}$$

The remaining gauge freedom is $\tilde{A} \to g^{-1}\tilde{A}g$ and $\tilde{B} \to g^{-1}\tilde{B}g$, where g is a constant matrix.

The invariance conditions

$$\mathcal{L}_X\Psi = \mathcal{L}_Y\Psi = 0 \tag{20}$$

imply that Ψ depends only on the invariant coordinates u, v and ζ. Taking into account (14) in the gauge where $A = B = 0$, the Lax pair (7)–(8) reduces to

$$\Psi_{,u} = \frac{1}{u-\lambda}\tilde{B}\Psi, \qquad \Psi_{,v} = \frac{1}{v-\lambda}\tilde{A}\Psi, \tag{21}$$

where we have set $\lambda = -\zeta^{-1}$.

Eqs. (18), (19) imply that

$$\partial_u\big(\operatorname{tr}\tilde{A}^k\big) = \partial_v\big(\operatorname{tr}\tilde{B}^k\big) = 0. \tag{22}$$

where $k = 1, 2, \ldots, N-1$. Hence

$$\operatorname{tr}\tilde{A}^k = m_k(v), \qquad \operatorname{tr}\tilde{B}^k = n_k(u). \tag{23}$$

In the following we require an autonomous reduction, i.e., the above first integrals m_k, n_k are chosen to be constant. Moreover, we restrict ourselves to the case where the gauge group is $\mathbf{GL}(2,\mathbb{C})$ and the Higgs fields $\tilde{A}$ and $\tilde{B}$ are singular matrix functions with eigenvalues $(0, m)$ and $(0, n)$, respectively, or, equivalently

$$\operatorname{rank}\big(\tilde{A}\big) = \operatorname{rank}\big(\tilde{B}\big) = 1.$$

In view of these algebraic constraints, $\tilde{A}, \tilde{B}$ may be written as

$$\tilde{A} = \begin{pmatrix} m - RQ & Q \\ R(m - RQ) & RQ \end{pmatrix},$$
$$\tilde{B} = \begin{pmatrix} n - PS & S \\ P(n - PS) & PS \end{pmatrix}. \tag{24}$$

Then, an obvious consequence of Eq. (19) is the relation $Q_{,u} = S_{,v}$ which implies the existence of a function $U(u,v)$ such that

$$Q = U_{,v}, \qquad S = U_{,u}.$$

The remaining equations that result from the substitution of (24) into (18) and (19) can easily be put in the following form:

$$P_{,v} = \frac{P-R}{u-v}\big(m + (P-R)U_{,v}\big), \tag{25}$$
$$R_{,u} = \frac{P-R}{u-v}\big(n - (P-R)U_{,u}\big), \tag{26}$$
$$U_{,uv} = \frac{1}{u-v}\big(nU_{,v} - mU_{,u} - 2(P-R)U_{,u}U_{,v}\big). \tag{27}$$

2.3. *The RPDE*

We may consider the functions P, R in system (25)–(27) as potentials for the function U. This consideration leads to a single fourth-order PDE as follows. First we solve (27) for the difference $P - R$ to get

$$P - R = \frac{1}{2}\left((v-u)\frac{U_{,uv}}{U_{,u}U_{,v}} + \frac{n}{U_{,u}} - \frac{m}{U_{,v}}\right). \tag{28}$$

Substituting this expression for $P-R$ into (25), we obtain $P_{,v}$ in terms of U and its derivatives only. Then, we differentiate (28) with respect to u and use (26) and (28) to eliminate $R_{,u}$ and $P-R$, respectively. This gives $P_{,u}$ in terms of derivatives of U. The compatibility condition $P_{,vu}=P_{,uv}$ leads to the following fourth-order PDE for U, which was called RPDE in [1]:

$$\begin{aligned}
&\mathcal{R}(u,v,U;m;n)\\
&\quad:= -U_{,uuvv}+U_{,uuv}\left(\frac{1}{u-v}+\frac{U_{,vv}}{U_{,v}}+\frac{U_{,uv}}{U_{,u}}\right)\\
&\qquad+U_{,uvv}\left(\frac{1}{v-u}+\frac{U_{,uu}}{U_{,u}}+\frac{U_{,uv}}{U_{,v}}\right)\\
&\qquad-U_{,uu}U_{,vv}\frac{U_{,uv}}{U_{,u}U_{,v}}\\
&\qquad+U_{,uu}\left(\frac{n^2}{(u-v)^2}\frac{U_{,v}^2}{U_{,u}^2}-\frac{1}{u-v}\frac{U_{,uv}}{U_{,u}}-\frac{U_{,uv}^2}{U_{,u}^2}\right)\\
&\qquad+U_{,vv}\left(\frac{m^2}{(u-v)^2}\frac{U_{,u}^2}{U_{,v}^2}+\frac{1}{u-v}\frac{U_{,uv}}{U_{,v}}-\frac{U_{,uv}^2}{U_{,v}^2}\right)\\
&\qquad+\frac{n^2}{2(u-v)^3}\frac{U_{,v}}{U_{,u}}\big(U_{,u}+U_{,v}+2(v-u)U_{,uv}\big)\\
&\qquad-\frac{m^2}{2(u-v)^3}\frac{U_{,u}}{U_{,v}}\big(U_{,u}+U_{,v}+2(u-v)U_{,uv}\big)\\
&\qquad+\frac{1}{2(u-v)}U_{,uv}^2\left(\frac{1}{U_{,u}}-\frac{1}{U_{,v}}\right)=0.
\end{aligned}\tag{29}$$

On the other hand, we may eliminate U from system (25)–(27). Solving the first two equations for the derivatives of U and substituting in (27) we get $U_{,uv}$ in terms of P, R and their derivatives. Differentiating (25) with respect to u and (26) with respect to v and expressing all the derivatives of U in terms of P, R and their derivatives, we get the following second-order coupled system for P and R:

$$P_{,uv}=\frac{2}{P-R}P_{,u}P_{,v}+\frac{m}{v-u}P_{,u}+\frac{n+1}{v-u}P_{,v},\tag{30}$$

$$R_{,uv}=\frac{2}{R-P}R_{,u}R_{,v}+\frac{m+1}{u-v}R_{,u}+\frac{n}{u-v}R_{,v}.\tag{31}$$

To decouple this system we solve Eq. (30) for R and substitute the result into (31). This yields a fourth-order PDE for P, namely

$$\mathcal{R}(u,v,P;m;n+1)=0.\tag{32}$$

In a similar way, solving Eq. (31) for P and substituting the result into (30) we get

$$\mathcal{R}(u,v,R;m+1;n)=0.\tag{33}$$

Thus, system (25)–(27) may be regarded as an involution of the RPDE $\mathcal{R}(u,v,U;m;n)=0$. Starting with a solution $U(u,v)$ of the RPDE, the system determines two solutions of the same equation but with the parameters m and n shifted by 1, respectively.

This shift in the parameters m and n which connects a triad of solutions of the RPDE may be considered as a remnant of the lattice systems that played a crucial role in the original derivation of the RPDE and its Miura associates — the MPDE and the SPDE — by Nijhoff et al. [1,5]. In that context, the parameters m and n represent the lattice variables and, as such, they are constrained to vary by unit steps only.

3. Auto-Bäcklund transformation

Using the Lax pair (21) one can derive an auto-Bäcklund transformation for system (25)–(27) by a relatively simple technique. Since the details of this derivation [6] are not pertinent to the present discussion, we prefer to present the result in the form of the following

Proposition 1. *The algebro-differential system*

$$\tilde{U}_{,u}=\frac{P-\tilde{U}}{u-\lambda}\big(n-(P-\tilde{U})U_{,u}\big),\tag{34}$$

$$\tilde{U}_{,v}=\frac{R-\tilde{U}}{v-\lambda}\big(m-(R-\tilde{U})U_{,v}\big),\tag{35}$$

$$(\tilde{U}-P)(U-\tilde{P})=u-\lambda,\tag{36}$$

$$(\tilde{U}-R)(U-\tilde{R})=v-\lambda\tag{37}$$

constitutes an auto-Bäcklund transformation for system (25)–(27).

Proof. The compatibility condition $\tilde{U}_{,uv}=\tilde{U}_{,vu}$ of (34), (35) leads to a second degree polynomial for $\tilde{U}$. This polynomial vanishes for every $\tilde{U}$ if and only if P,R,U satisfy system (25)–(27).

Conversely, solving Eqs. (34), (35) with respect to the derivatives of U and using the algebraic relations

(36), (37), we get the system

$$U_{,u} = \frac{\tilde{P} - U}{u - \lambda}\big(n - (\tilde{P} - U)\tilde{U}_{,u}\big), \tag{38}$$

$$U_{,v} = \frac{\tilde{R} - U}{v - \lambda}\big(m - (\tilde{R} - U)\tilde{U}_{,v}\big). \tag{39}$$

The compatibility condition holds for every U iff $\tilde{P}, \tilde{R}, \tilde{U}$ satisfy system (25)–(27).

The algebraic relations (36), (37) are compatible with system (25)–(27) in the following sense. Their differential consequences lead to Eqs. (25), (26) for $\tilde{P}, \tilde{R}$. To prove this, we first express $\tilde{P}, \tilde{R}$ in terms of $U, P, R, \tilde{U}$:

$$\tilde{P} = U + \frac{u - \lambda}{P - \tilde{U}}, \qquad \tilde{R} = U + \frac{v - \lambda}{R - \tilde{U}}. \tag{40}$$

Differentiating these equations with respect to v and u, respectively, we get

$$\tilde{P}_{,v} = U_{,v} - \frac{u - \lambda}{(P - \tilde{U})^2}\big(P_{,v} - \tilde{U}_{,v}\big),$$

$$\tilde{R}_{,u} = U_{,u} - \frac{v - \lambda}{(R - \tilde{U})^2}\big(R_{,u} - \tilde{U}_{,u}\big). \tag{41}$$

We now use Eqs. (25), (26) to eliminate the derivatives of P, R and (38), (39) to substitute the derivatives of U. Using (36), (37) to replace P, R, we end up with the equations

$$\tilde{P}_{,v} = \frac{\tilde{P} - \tilde{R}}{u - v}\big(m + (\tilde{P} - \tilde{R})\tilde{U}_{,v}\big), \tag{42}$$

$$\tilde{R}_{,u} = \frac{\tilde{P} - \tilde{R}}{u - v}\big(n - (\tilde{P} - \tilde{R})\tilde{U}_{,u}\big). \tag{43}$$

These are identical to Eqs. (25), (26) with P, R, U replaced by $\tilde{P}, \tilde{R}, \tilde{U}$. Conversely, if our starting point are the functions $\tilde{P}, \tilde{R}, \tilde{U}$ then the differential consequences of Eqs. (36), (37) will be Eqs. (25), (26) for P, R and U. □

One may now easily prove the following permutability theorem. It is worth noting that the superposition principle is very simple and remarkably similar to the one connecting four solutions of the KdV equation.

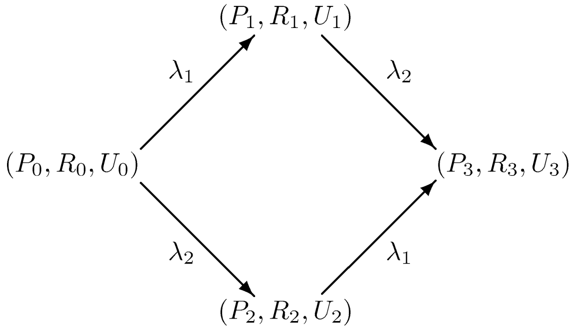

Fig. 1. Bianchi commuting diagram.

Permutability theorem. *Let* (U_i, P_i, R_i), $i = 1, 2$, *be a solution of system (*25)–(27)*, generated by means of the Bäcklund of transformation* (34)–(37) *from a known solution* (U_0, P_0, R_0) *via the Bäcklund parameters* λ_1 *and* λ_2*, respectively. Then there exists a new solution* (U_3, P_3, R_3) *which is given by*

$$(U_3 - U_0)(U_2 - U_1) = \lambda_2 - \lambda_1, \tag{44}$$

$$(P_3 - P_0)(P_2 - P_1) = \lambda_2 - \lambda_1, \tag{45}$$

$$(R_3 - R_0)(R_2 - R_1) = \lambda_2 - \lambda_1, \tag{46}$$

where (U_3, P_3, R_3) *is constructed according to Fig.* 1.

The auto-Bäcklund transformation (34)–(37) produces an auto-Bäcklund of the RPDE in the following fashion. Beginning with a solution of the RPDE $\mathcal{R}(u, v, U; m; n) = 0$, we may calculate the functions P, R using system (25)–(27). Then, the integration of the Riccati system (34), (35) yields a new solution $\tilde{U}$ of the same equation.

4. Symmetry analysis and group invariant solutions of the RPDE

In [1] it was shown that the RPDE is related to the KdV hierarchy. In a subsequent paper [6] we shall show that the solution space of the Ernst–Weyl equations describing the collision of two neutrino waves accompanied by gravitational waves is embedded in the solution space of the RPDE. Hence, it is interesting to consider specific solutions of the RPDE. Of particular interest are the so-called group invariant solutions [7–9]. In this section we first calculate the symmetry algebra of the RPDE. Then, the group invariant solutions, for each of the one-dimensional subgroups in the optimal system, are explicitly constructed.

4.1. Lie point symmetries

An infinitesimal symmetry of the RPDE is represented by the vector field

$$\mathbf{v} = \xi(u,v,U)\partial_u + \zeta(u,v,U)\partial_v + \phi(u,v,U)\partial_U. \tag{47}$$

Using the fourth-order prolongation of $\mathbf{v}$ [7,8] and taking into account the RPDE itself, we get a linear overdetermined system of PDE's for ξ, ζ and ϕ. The determining equations form a very large system, which was obtained using SPDE for REDUCE [10] and MATHLIE for MATHEMATICA [11]. The six simplest equations of this system are

$$\xi_{,v} = \xi_{,U} = \zeta_{,u} = \zeta_{,U} = \phi_{,u} = \phi_{,v} = 0, \tag{48}$$

and these reduce the rest of the determining equations to the following simple system:

$$(u-v)\xi_{,u} + \zeta - \xi = 0, \tag{49}$$

$$(u-v)\zeta_{,v} + \zeta - \xi = 0, \tag{50}$$

$$\phi_{,UUU} = 0. \tag{51}$$

The general solution of this system is given by

$$\xi(u) = \alpha u + \beta, \qquad \zeta(v) = \alpha v + \beta,$$
$$\phi(U) = \gamma_1 + \gamma_2 U + \gamma_3 U^2. \tag{52}$$

Therefore, the symmetry algebra $\mathfrak{g}$ of the RPDE is five-dimensional, spanned by the following vector fields:

$$\mathbf{v}_1 = \partial_u + \partial_v, \qquad \mathbf{v}_2 = u\partial_u + v\partial_v,$$
$$\mathbf{v}_3 = \partial_U, \qquad \mathbf{v}_4 = U\partial_U, \qquad \mathbf{v}_5 = U^2\partial_U. \tag{53}$$

The generators $\{\mathbf{v}_1, \mathbf{v}_2\}$ form a two-dimensional non-Abelian solvable Lie algebra $\mathfrak{h}$. The subalgebra spanned by $\{\mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ is isomorphic to $\mathfrak{sl}(2)$. Thus,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{sl}(2). \tag{54}$$

The group generated by $\mathbf{v}_1, \mathbf{v}_2$ consists of base transformations, i.e., transformations that leave the dependent variable unaffected,

$$(u,v,U) \to (\alpha u + \beta, \alpha v + \beta, U), \quad \alpha, \beta \in \mathbb{R}. \tag{55}$$

In contrast, the remaining vector fields generate the projective group $PSL(2,\mathbb{C})$ acting only on U (Möbius transformation):

$$(u,v,U) \to \left(u, v, \frac{\alpha U + \beta}{\gamma U + \delta}\right),$$
$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL(2,\mathbb{C}). \tag{56}$$

One can now classify the subalgebras of the symmetry algebra $\mathfrak{g}$ very easily. Since $\mathfrak{g}$ is the direct sum of $\mathfrak{h}$ and $\mathfrak{sl}(2)$, the problem reduces to the classification of the subalgebras of these two algebras. The optimal system [7, p. 203] of one-dimensional subalgebras of $\mathfrak{h}$ is two-dimensional and is generated by $\mathbf{v}_1$ and $\mathbf{v}_2$. The corresponding optimal system of $\mathfrak{sl}(2)$ is two-dimensional and is generated by the vector fields $\mathbf{v}_3$ and $\mathbf{v}_4$. Thus, the optimal system of one-dimensional subalgebras of the RPDE is spanned by

$$\mathbf{v}_1,\ \mathbf{v}_1 + \mu\mathbf{v}_3,\ \mathbf{v}_1 + \mu\mathbf{v}_4,$$
$$\mathbf{v}_2,\ \mathbf{v}_2 + \mu\mathbf{v}_3,\ \mathbf{v}_2 + \mu\mathbf{v}_4, \quad \mu \neq 0. \tag{57}$$

4.2. Reductions to the Schwarzian equation

The group invariant solutions under the vector fields $\mathbf{v}_1, \mathbf{v}_2$ are solutions of the RPDE of the form

$$U(u,v) = F(u-v), \qquad U(u,v) = F\left(\frac{u}{v}\right), \tag{58}$$

respectively. Inserting these into the RPDE and integrating once the reduced ODE, for each case, results in the Schwarzian equation

$$\{F, y\} = \frac{F_{,yyy}}{F_{,y}} - \frac{3}{2}\frac{F_{,yy}^2}{F_{,y}^2} = G(y), \tag{59}$$

where

$$G(y) = \begin{cases} \frac{\kappa}{y^2}, \quad y = u - v, \quad \text{for } \mathbf{v}_1, \\ \frac{n^2}{2(y-1)^2} + \frac{m^2}{2y^2(y-1)^2} - \frac{1}{2y^2} + \frac{\kappa}{2y(y-1)^2}, \\ \quad y = u/v, \quad \text{for } \mathbf{v}_2, \end{cases} \tag{60}$$

and κ a constant of integration. The general solution of (59) has the form $F = \varphi(y)/\psi(y)$, where $\varphi(y)$ and $\psi(y)$ are two arbitrary, linearly independent solutions of the linear Schrödinger equation $\psi_{,yy} + (1/2) \times G(y)\psi = 0$ [8,12].

4.3. Reduction to Painlevé III

The invariant solutions of the RPDE under the vector field $\mathbf{v}_1 + \mu \mathbf{v}_3$ satisfy the differential constraint

$$\mu - U_{,u} - U_{,v} = 0. \tag{61}$$

Thus, $U(u, v)$ is of the form

$$U(u, v) = F(u - v) + \frac{1}{2}\mu(u + v). \tag{62}$$

Substituting this form into the RPDE, we get a fourth-order ODE for $F(y)$, where $y = u - v$. The order of this ODE may be reduced by two on setting

$$F'(y) = \frac{\mu}{2}\frac{1 + G(y)}{1 - G(y)} \tag{63}$$

and integrating once. The reduced second-order ODE for $G(y)$ is Painlevé V with $\delta = 0$ (compare with Eq. (70) below), i.e.,

$$G'' = \left(\frac{1}{2G} + \frac{1}{G-1}\right)G'^2 - \frac{1}{y}G' + \frac{n^2}{2}\frac{G(G-1)^2}{y^2} - \frac{m^2}{2}\frac{(G-1)^2}{y^2 G} + \kappa\frac{G}{y}, \tag{64}$$

where κ is the constant of integration. In this case, Painlevé V reduces to Painlevé III [13] in full form,[1]

$$v''(z) = \frac{v'(z)^2}{v(z)} - \frac{v'(z)}{z} + \frac{\tilde{\alpha} v(z)^2 + \tilde{\beta}}{z} + v(z)^3 + \frac{\kappa^2}{v(z)}, \tag{65}$$

through the transformation

$$z^2 = 2y, \qquad u(z) = \frac{1 + G(z)}{1 - G(z)}, \tag{66}$$

$$u(z) = \frac{v'(z)}{v(z)^2} - \frac{\tilde{\beta} + \kappa}{\kappa z v(z)} + \frac{\kappa}{v(z)^2}, \tag{67}$$

$$\left(1 - u(z)^2\right)v(z) = u'(z) + \left(2 + \frac{\tilde{\beta}}{\kappa}\right)\frac{u(z)}{z} - \frac{\tilde{\alpha}}{z}. \tag{68}$$

The parameters $\tilde{\alpha}, \tilde{\beta}$ are related to the parameters α, β of the Painlevé V through

$$2\alpha = n^2 = \frac{((\tilde{\alpha} + 2)\kappa + \tilde{\beta})^2}{16\kappa^2},$$

$$-2\beta = m^2 = \frac{(\tilde{\beta} - (\tilde{\alpha} - 2)\kappa)^2}{16\kappa^2}. \tag{69}$$

4.4. Reduction to Painlevé V

The group invariant solutions of the RPDE for the vector field $\mathbf{v}_1 + \mu \mathbf{v}_4$ are related to Painlevé V in full form, i.e., to the ODE

$$G'' = \left(\frac{1}{2G} + \frac{1}{G-1}\right)G'^2 - \frac{1}{y}G' + \alpha\frac{G(G-1)^2}{y^2} + \beta\frac{(G-1)^2}{y^2 G} + \gamma\frac{G}{y} + \delta\frac{G(G+1)}{G-1}. \tag{70}$$

The invariant solution under this generator satisfies the PDE

$$\mu U - U_{,u} - U_{,v} = 0, \tag{71}$$

which has the general solution

$$U(u, v) = F(u - v)\exp\left(\frac{1}{2}\mu(u + v)\right). \tag{72}$$

The RPDE becomes now a fourth-order ODE which, after the substitution

$$F'(y) = \frac{\mu}{2}\frac{1 + G(y)}{1 - G(y)}F(y), \quad y = u - v, \tag{73}$$

can be integrated once leading to the full Painlevé V (70) with parameters

$$\alpha = \frac{n^2}{2}, \qquad \beta = -\frac{m^2}{2}, \qquad \gamma = \kappa, \qquad \delta = -\frac{\mu^2}{2}, \tag{74}$$

where κ is the constant of integration.

4.5. Reductions to Painlevé VI

The remaining generators of the optimal system lead to reductions related to Painlevé VI. The procedure is the same as in the previous cases. First, we integrate the first-order differential constraint leading to a specific form for the function $U(u, v)$, in terms of an unknown function F which depends on $y = u/v$. The substitution into the RPDE leads to a fourth-order ODE which, after an appropriate substitution for the first derivative of F, is integrated once leading to

[1] Using a scaling transformation one may always set one of the parameters of the Painlevé III equal to 1.

Table 1

Symmetry generator and differential constraint	Substitution and quadrature	Parameters in P_{VI} for $G(y)$
$\mathbf{v}_2+\mu\mathbf{v}_3$, $\mu-uU_{,u}-vU_{,v}=0$	$U(u,v)=F(y)+\frac{\mu}{2}\log(uv)$, $F'(y)=\frac{\mu}{2y}\frac{y+G(y)}{y-G(y)}$	$\alpha=\frac{n^2}{2}$, $\beta=-\frac{m^2}{2}$, $\gamma=\frac{1}{2}(m^2+n^2+\kappa)$, $\delta=\frac{1}{2}$
$\mathbf{v}_2+\mu\mathbf{v}_4$, $\mu U-uU_{,u}-vU_{,v}=0$	$U(u,v)=F(y)(uv)^{\mu/2}$, $F'(y)=\frac{\mu}{2y}\frac{y+G(y)}{y-G(y)}F(y)$	$\alpha=\frac{n^2}{2}$, $\beta=-\frac{m^2}{2}$, $\gamma=\frac{1}{2}(m^2+n^2+\mu^2+\kappa)$, $\delta=\frac{1-\mu^2}{2}$

Painlevé VI,

$$G''=\frac{1}{2}\left(\frac{1}{G}+\frac{1}{G-1}+\frac{1}{G-y}\right)G'^2 -\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{G-y}\right)G' +\frac{G(G-1)(G-y)}{y^2(y-1)^2} \times\left(\alpha+\beta\frac{y}{G^2}+\gamma\frac{y-1}{(G-1)^2}+\delta\frac{y(y-1)}{(G-y)^2}\right). \tag{75}$$

The results are listed in Table 1.

The first case, where full Painlevé VI occurred as a reduction of an integrable scalar PDE, was presented in [1]. It resulted from the reduction of the SPDE and, since the latter is related to the RPDE through a chain of Miura maps [1], the reduction of the RPDE itself to Painlevé VI, as presented above, was a well expected result.

5. Perspectives

It is well known that reductions of the ASDYM equations lead to physically interesting integrable equations by imposing appropriate reality conditions. In this Letter we have restricted to a real slice without imposing any further conditions on the, generally, complex dependent variables of system (25)–(27). In a subsequent paper [6] we shall show that the Ernst–Weyl equation, describing the collision of neutrino waves accompanied by gravitational waves [14], arises naturally within this framework. From this standpoint, it is justified to view the RPDE as a generalization of the Ernst–Weyl equation. The latter contains the famous Ernst equation when the neutrino fields vanish everywhere. Our reduction scheme unifies many aspects of the integrability of the Ernst equation like the Hauser–Ernst Lax pair, Neugebauer–Kramer involution and the Harrison Bäcklund transformation and gives analogous generalizations for the Ernst–Weyl equation.

With regard to the above mentioned relation between the RPDE and the Ernst–Weyl equation, we would like to point out that the results of the last section link up with the recent work of Schief [15], who demonstrated that the Ernst–Weyl equation for axially symmetric spacetimes reduces to the full Painlevé III, V and VI. The relevant advantage of our approach is that it enables one to produce such reductions in a straightforward manner from those of the RPDE, namely by restricting the solution space of the latter.

In this Letter we restricted our considerations to the first member of the Miura chain of PDEs consisting of the RPDE, the MPDE and the SPDE. It remains to be seen if the other two members of the chain can also be obtained by a two-dimensional reduction of the ASDYM equations, analogous to the one that led to the RPDE. The derivation of the Lie point symmetries of the MPDE and the SPDE and the construction of similarity solutions in the manner presented above for the RPDE are equally interesting projects. In this respect, it is of particular interest to consider the relation between the Miura transformations connecting the RPDE with the MPDE and the SPDE [1], on the one hand, and analogous transformations connecting the ODEs obtained by similarity reduction of the corresponding PDES, on the other. The results of Nijhoff et al. [5] on the Miura chain connecting the (regular) Painlevé VI with its modified and Schwarzian counterparts are expected to play a significant role in clarifying this issue.

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